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Aspects of Proficiency in School Algebra

(Symbol Sense: Arcavi 1994)

making sense of...

...elements of algebra

knowing

to identify or reproduce important rules and technical terms

acting

transformational types of activities (Kieran 2004)

generational types of activities (Kieran 2004)

structuring

to transform or interpret an algebraic expression while maintaining its structure (substitutional equivalence: Musgrave et al. 2005; Rüede 2015)

transforming

to transform an algebraic expression into an equivalent expression of different structure (transformational equivalence: Musgrave et al. 2005; treatment: Duval 2006)

interpreting

to describe a non-algebraic situation by formal algebra and vice versa (conversion: Duval 2006)

variables incl. parameters

Variables are signs that represent numbers or quantities. Parameters are variables that vary over sets of values of other variables (Veränderliche vs. Einzelzahl: Malle 1993, Variable vs. Metavariable: Drijvers 2001, values taken by a variable: Bardini et al. 2005). This discriminations arises from the context of the task.

no meaningful aspects

(3) to recognize applicabilty of transformation rules

An expression is identified as a representation of a class of structurally equivalent expressions and rules of transformation that are associated with this class. This is done by, mentally or explicitely, substituting variables or terms by terms or variables (systemic structure: Kieran 1989; structure sense: Hoch & Dreyfus 2006)

(4) to recognize the operational ordering

The logical ordering of the operations within an expression is recognized. This is done by, mentally or explicitely, substituting terms by variables (surface structure, Kieran 1989; Rechenschema: Vollrath & Weigand 1993; Rechenhandlung: Malle 1993)

(7) to interprete variables and parameters

Variable signs are interpreted or used as representations of numbers (Einsetzungs-, Gegenstandsaspekt: Malle 1993, Küchemann 1978). Within given contexts, appropriate variables are identified or used as parameters.

(10) to switch between expressions and real situations

An expression or equation is translated to a real-life situation, and vice versa (McGregor & Stacey 1995; Heid 1996), e.g. when viewed as a function (Nitsch 2015). This activity involves a higher gradient of abstraction than activities (7,8,9) that results from the need to replace the inevitable concrete realistic mental model of the given real situation by an abstract-formal mental model before formulating an expression (Malle 1993)

expressions and equations

Algebraic expressions are compositions of variables and arithmetic operation signs. When a variable is viewed as representing a range of number values or quantitites (variable object: Schoenfeld & Arcavi 1988; Bereichsaspekt: Malle 1993) the value of the expression is interpreted as a function of this variable (Malle 1993; Heid 1996).

Equations are expressions where two terms are compared with regard to their values, symbolized by an equation sign. An equation differs from a computation or transformation of a term in that it is used in a relational sense (notion of equivalence: Kieran 1981; operational vs. relational view: Baroody & Ginsburg 1983; Zuweisungs- vs. Vergleichszeichen: Malle 1993).

(1) to specify transformation rules or terminology

Important technical terms for expressions and rules for manipulating expressions or equations are identified or specified, e.g. names for classes of terms or equations, or rules for simplifying expressions, binomial rules, rules for solving quadratic equations,etc.

(2) to transform following given rules

Expressions and equations are transformed into equivalent expressions or equations by applying given rules (manipulation skills: Hoch & Dreyfus 2006)

(5) to compute or to compare

An expression with an equation sign is interpreted in an operational or a relational sense, as it is appropriate in the context (Malle 1993; operational vs. relational view: Baroody & Ginsburg 1983; Knuth, Alibali & al. 2006).

(8) to switch between expressions and innermathematical situations

A non-algebraic but innermathematical situation (e.g. dot patterns or geometric configurations) is described by a term or an equation, and vice versa (Bauplan: Vollrath & Weigand 2009).

(6) to transform (efficiently)

Expressions and equations are being transformed into equivalent expressions or equations (2,4), by activating existing knowledge about transformation rules (1) which are identified as applicable to the present problem (3). Also, two expressions or equations are identified as equivalent „on the spot" without applying rules explicitely (algebraic expectation, Pierce & Stacey 2001). A transformation is „efficient" if, among various rules of transformation that are applicable, one is chosen that allows relatively few steps and few computations (strategic flexibility: Star & Rittle-Johnson 2009; vgl. auch Malle 1993, S.188ff.).

(9) to switch between expressions and tables or graphs

An expression or equation is translated to a value table or a graph, and vice versa (McGregor & Stacey 1995), e.g. when viewed as a function (Duval 2006, Nitsch 2015), or for solving an equation (Arcavi 1994)

Whichof the following are correct exponent laws?

$(a \cdot b)^p = a^p + b^p$
$(a + b)^p = a^p \cdot b^p$
$(a + b)^p = a^p + b^p$
$(a \cdot b)^p = a^p \cdot b^p$

In an unusual number range, three times a number is the number itself. I.e., for any number x, 3x = x holds. Any other law for adding or multiplying remains unchanged.

Simplify each of the following expressions while applying this rule:

3a =   3b – b =
5·2 =   9 =

Factorize:

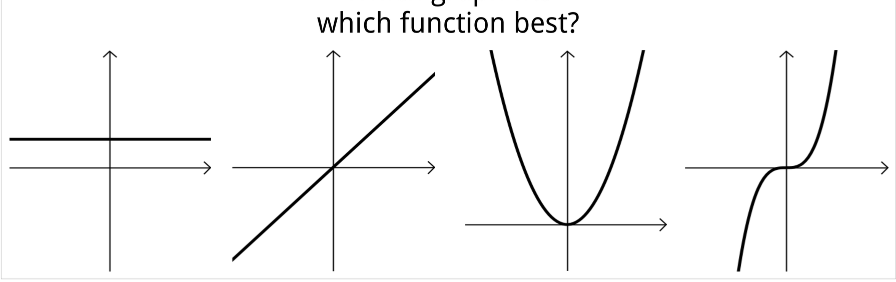
$16 - x^2 =$ ______

You do not need to solve the equation $7(x-2) = 3(x-2) + 16$

But what would be your first step?

Three functions are given:
$f(s) = s \cdot t^2$, $g(t) = s \cdot t^2$, and $h(x) = s \cdot t^2$

Which graph fits which function best?

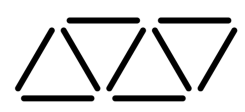


Fill in the blanks:
3a + 2a = ____ + 4a = ____

The table shows the values of function f

x	-2	-1	0	1	2	3
f(x)	9	4	1	0	1	4

Determine an equation of f.



Determine a formula for the number of matches needed for making k triangles.

Aaron is a cm tall,
Berta is b cm tall.
Berta is 10 cm smaller than Aaron.
Give an equation that describes how a and b are related.

What?

A comprehensive and summative overview of aspects of formal school algebra, focussing on • **algebra, not functions**: a model about algebraic profiency must cover some, but cannot cover all aspects of the concept of function. So, functions are only present when being the result of a functional interpretation of an algebraic expression • **formal, not generic:** at the end of secondary school maths, a student's proficiency in algebra must have reached a stage of being competent with symbolic representations of indeterminate number values and quantities and relations between them. So the model is restricted to aspects of formal algebra • **summative, not formative:** the model is meant to comprise all important aspects of profiency at the end of secondary school maths, not while they are being taught. Thus it is meant to be a conceptual frame for summative diagnosis, not formative.

Why?

For being successful in STEM subjects at high school or university, a good mastery of formal algebra is indispensable. But what does that mean? **To have a clear position in discussions between maths educators from school and university**, the school perspective needs to have a comprehensive and systematic overview on the various aspects of proficiency in school algebra. At the same time, the model serves as a conceptual frame for a summative diagnosis at the transit from school to university.

How?

Relevant positions and findings from educational research were categorized along two a priori dimensions, which are "making sense of..." and "...elements of algebra". The categories are accompanied by about 70 tasks which were compiled from existing sources or created new. Both model and tasks were presented to experts for validation.

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